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Topological and Non-topological Solitons for the Generalized Zakharov-Kuznetsov Modified Equal Width Equation

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Abstract This paper obtains the topological and non-topological solitary wave solution of the generalized Zakharov-Kuznetsov modified equal width equation. The solitary wave ansatz method is used to carry out the integration of this equation. A couple of conserved quantities are calculated for the non-topological solitons. The domain restriction is identified for the power law nonlinearity parameter.

Keywords Solitons · Integrals of motion · Integrability

1 Introduction

The study of nonlinear evolution equations has been going on for the past few decades [1–15]. There has been a large number of equations that are studied in this context here. One of the major questions that arise in the study of these equations is its integrability. In particular, the thirst is to seek soliton solutions of such equations. There are many newly developed techniques to solve these newly generated nonlinear evolution equations in its generalized forms [8, 9, 11]. These newly developed techniques are a true blessing in this area of Applied Mathematics.

In this paper, there is one such important equation that will be studied. It is called the generalized Zakharov-Kuznetsov modified equal width equation (gZK-MEW). There are various kinds of solutions that are already available for this equation. They are the singular solutions, compactons, periodic solutions, cnoidal waves and others. In this paper, the focus is going to be obtaining a 1-soliton solution to gZK-MEW equation. Both the topological as well as the non-topological solitary wave solutions will be obtained in this paper. The technique that will be used is the solitary wave ansatz method. It needs to be noted that this method of solitary wave ansatz is similar to the exponential function method. The advantage of this method solitary wave ansatz is that it can easily compute the soliton velocity which

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is not possible by the well-known travelling wave hypothesis. Also, the relation between the amplitude and the widths can also be obtained along with the domain restrictions of the soliton parameters. These are not possible in the travelling wave hypothesis method, for example. This soliton solution will also be used to compute the conserved quantity for this equation for the case of non-topological solitons.

2 Mathematical Analysis

The dimensionless form of the gZK-MEW equation that is going to be studied in this paper is given by [8, 14]

$$q_t + a\left(q^n\right)_x + \left(bq_{xt} + cq_{yy}\right)_x = 0 \tag{1}$$

Here in (1) a, b and c are real valued constants. The first term represents the evolution term while the second term is the nonlinear term and finally the third and fourth term together, in parenthesis, are the dispersion terms. The solitons are a result of a delicate balance between dispersion and nonlinearity. The exponent n, which indicates the power law nonlinearity parameter is such that $n > 1$. In order to obtain the 1-soliton solution of (1), the study will be split into the following two sub-sections which talks about the non-topological and topological solitons respectively.

2.1 Non-topological Solitons

The non-topological solitons are also known as the bell-shaped solitons. These are normally modelled by the sech functions. Based on this norm, the solitary wave ansatz for the 1-soliton solution of (1) is taken to be [8, 14]

$$q(x, y, t) = \frac{A}{\cosh^p\left(B_1 x + B_2 y - vt\right)} \tag{2}$$

where A represents the amplitude of the soliton, while B_1 and B_2 represents the inverse width in the x- and y-directions respectively. Also, v represents the velocity of the soliton and the exponent p, which is unknown at this point, will be determined in the course of derivation of the exact solution. It is to be noted that in the hypothesis of the solution structure, the inverse widths of the soliton in the x- and y-directions are taken to be different, namely $B_1 \neq B_2$, in general. This makes the structure of the soliton solution generalized. From (2), using the notation

$$\tau = B_1 x + B_2 y - vt \tag{3}$$

it is possible to get

$$q_t = pvA\frac{\tanh\tau}{\cosh^p\tau} \tag{4}$$

$$\left(q^n\right)_x = -npA^n B_1\frac{\tanh\tau}{\cosh^{np}\tau} \tag{5}$$

$$q_{xxt} = p^3 vAB_1^2\frac{\tanh\tau}{\cosh^p\tau} - p(p+1)(p+2)vAB_1^2\frac{\tanh\tau}{\cosh^{p+2}\tau} \tag{6}$$

and

$$q_{yyx} = -p^3 A B_2^2 B_1 \frac{\tanh\tau}{\cosh^p\tau} + p(p+1)(p+2) A B_2^2 B_1 \frac{\tanh\tau}{\cosh^{p+2}\tau} \tag{7}$$

Substituting (4)–(7) into (1), gives

$$\begin{aligned} &pvA\frac{\tanh\tau}{\cosh^p\tau} - anpA^n B_1 \frac{\tanh\tau}{\cosh^{np}\tau} \\ &\quad + b\left\{p^3 vAB_1^2 \frac{\tanh\tau}{\cosh^p\tau} - p(p+1)(p+2)vAB_1^2\frac{\tanh\tau}{\cosh^{p+2}\tau}\right\} \\ &\quad - c\left\{p^3 AB_2^2 B_1 \frac{\tanh\tau}{\cosh^p\tau} - p(p+1)(p+2)AB_2^2 B_1 \frac{\tanh\tau}{\cosh^{p+2}\tau}\right\} = 0 \end{aligned} \tag{8}$$

Now, equating the exponents np and $p+2$ gives

$$np = p + 2 \tag{9}$$

that results in

$$p = \frac{2}{n-1} \tag{10}$$

Again, noting that the functions $\tanh\tau/\cosh^p\tau$ and $\tanh\tau/\cosh^{p+2}\tau$ are linearly independent, setting their respective coefficients in (8) to zero yields

$$v = \frac{4cB_1B_2^2}{(n-1)^2 + 4bB_1^2} \tag{11}$$

and

$$A = \left[\frac{2c(n+1)B_2^2}{a}\right]^{\frac{1}{n-1}} \tag{12}$$

Thus, the 1-soliton solution of the gZK-MEW equation, with power law nonlinearity, is given by

$$q(x,y,t) = \frac{A}{\cosh^{\frac{2}{n-1}}(B_1x + B_2y - vt)} \tag{13}$$

where the amplitude A is related to the inverse widths B_1 and B_2 as given by (12) and the velocity v is given by (11).

2.1.1 Integrals of Motion

The gZK-MEW equation, with power law nonlinearity, permits at least two conserved quantities [6]. They are the linear momentum (P) and energy (E) that are respectively given by

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(x,y,t)\,dxdy = \frac{A}{B_1B_2}\frac{\Gamma(\frac{1}{n-1})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n-1}+\frac{1}{2})} \tag{14}$$

and

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q^2(x,y,t)dxdy = \frac{A^2}{B_1 B_2}\frac{\Gamma(\frac{2}{n-1})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{n-1}+\frac{1}{2})} \tag{15}$$

where $\Gamma(x)$ is the gamma function that is defined as

$$\Gamma(x) = \int_{-\infty}^{\infty} e^{-u}u^{x-1}du \tag{16}$$

Thus from the domain of gamma functions, it is possible to say from (14) and (15) that solitons for (1) will exist for $n > 1$.

2.2 Topological Solitons

Topological solitons are stable, particle-like localised objects, with finite mass and a smooth structure. They also superimpose nonlinearly and retain their identity after interaction. The topology of a soliton is specified by one or two integers that are known as topological charges which Mathematically specify the homotopy class or characteristic class. A simplest example of a topological soliton is a *kink* that lives on the real line [11].

For topological solitons, the solitary wave ansatz for the 1-soliton solution of (1) is taken to be [13, 14]

$$q(x,y,t) = A\tanh^p(B_1x + B_2y - vt) \tag{17}$$

where A, B_1 and B_2 are free parameters and v represents the velocity of the soliton. The exponent p, which is unknown at this point, will be determined in the course of derivation of the exact solution. From (17) it is possible to obtain

$$q_t = -pvA\left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right) \tag{18}$$

$$(q^n)_x = npA^nB_1\left(\tanh^{np-1}\tau - \tanh^{np+1}\tau\right) \tag{19}$$

$$\begin{aligned} q_{xxt} = -pvAB_1^2\big[&(p-1)(p-2)\tanh^{p-3}\tau \\ &- \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau + \left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau \\ &- (p+1)(p+2)\tanh^{p+3}\tau\big] \end{aligned} \tag{20}$$

and

$$\begin{aligned} q_{yyx} = -pAB_1^2B_2\big[&(p-1)(p-2)\tanh^{p-3}\tau \\ &- \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau + \left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau \\ &- (p+1)(p+2)\tanh^{p+3}\tau\big] \end{aligned} \tag{21}$$

Substituting (18)–(21) into (1), gives

$$\begin{aligned} &-pvA\left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right) + anpA^nB_1\left(\tanh^{np-1}\tau - \tanh^{np+1}\tau\right) \\ &\quad - bpvAB_1^2\big[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau \\ &\quad + \left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\big] \end{aligned}$$

$$+ cpAB_1^2B_2\left[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau \right.$$
$$\left. + \left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\right] \tag{22}$$

Now, equating the exponents $np+1$ and $p+3$ gives

$$np+1 = p+3 \tag{23}$$

that gives the same value of p as in (9). This is also obtained when the exponents $np-1$ and $p+1$ are equated. Again, noting that the functions $\tanh^{p+j}\tau$ are linearly independent for $j = -3, -1, 1, 3$, setting their coefficients to zero yields:

$$v = \frac{2cB_1B_2^2(n^2-5n+10)}{2bB_1^2(n^2-5n+10)-(n-1)^2} \tag{24}$$

and

$$v = \frac{2cB_1B_2^2(n^2+n+4) - an(n-1)A^{n-1}B_1}{1-2bB_1^2(n^2+n+4)} \tag{25}$$

Now equating the two values of the velocity (v) of the soliton yields the relation between the free parameters A, B_1 and B_2 as

$$A = \left(\frac{M}{N}\right)^{\frac{1}{n-1}} \tag{26}$$

where

$$M = 2cB_2^2\left\{(n-1)^2\left(n^2+4n+4\right) + \left(n^2-5n+10\right)\right.$$
$$\left. - 4bB_1^2\left(n^2-5n+10\right)\left(n^2+4n+4\right)\right\} \tag{27}$$

and

$$N = an(n-1)\left\{(n-1)^2 - 2bB_1^2\left(n^2-5n+10\right)\right\} \tag{28}$$

The velocity of the soliton is also given by

$$v = \frac{cB_2^2}{bB_1} \tag{29}$$

which is obtained from setting the coefficients of $\tanh^{p-3}\tau$ or $\tanh^{p+3}\tau$ to zero. Thus, the topological 1-soliton solution of the gZK-MEW equation, with power law nonlinearity, is given by

$$q(x,y,t) = A\tanh^{\frac{2}{n-1}}(B_1x+B_2y-vt) \tag{30}$$

where the relation between A, B_1 and B_2 are given in (26) and the velocity v is given by (29). It needs to be noted from (26) or (30) that it necessary to have $n \neq 1$.

3 Conclusions

This paper obtains both topological and non-topological soliton solution of the gZK-MEW equation. The solitary wave ansatz is used to carry out the integration of this equation. It is observed that there is no expression for the soliton radiation in this paper as this technique fails to obtain soliton radiation. In future, the perturbations terms of this equation will be considered and will be studied using the soliton perturbation theory. In addition to the deterministic perturbation terms, stochastic perturbation terms will be considered too.

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